Blow-up Solutions to a Viscoelastic Fluid System and a Coupled Navier-Stokes/Phase-Field System in $\mathbb{R}^{2*}$

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The explicit solutions to both the Oldroyd-B model with infinite Weissenberg number and the coupled Navier-Stokes/phase-field system are constructed by the method of separation of variables. It is found that the solutions blow up in finite time.

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The Oldroyd-B model with infinite Weissenberg number for an incompressible viscoelastic fluid system in $\mathbb{R}^n$ takes the form (see [1, 2]):

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\Delta\mathbf{u} + \lambda\nabla\cdot(\mathcal{F}\mathcal{F}^T), \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

$$\mathcal{F}_t + \mathbf{u}\cdot\nabla\mathcal{F} = \nabla\mathbf{u}\mathcal{F}, \tag{3}$$

where $\mathbf{u}$ represents the velocity vector in $\mathbb{R}^n$, p is a scalar function denoting the pressure and $\mathcal{F}$ is a $n\times n$ matrix denoting the deformation tensor. ν and λ are positive constants, denoting the kinetic viscosity and the competition between kinetic energy and elastic energy, respectively.

Let $\mathbf{x}$ be the Eulerian coordinates and $\mathbf{X}$ the Lagrangian coordinates. The flow map $\mathbf{x}(X,t)$ is defined by

$$\mathbf{x}_t(X,t) = \mathbf{u}(\mathbf{x}(X,t),t), \quad \mathbf{x}(X,0) = X.$$

The deformation tensor $\tilde{\mathcal{F}}(X,t) = \frac{\partial\mathbf{x}}{\partial\mathbf{X}}(X,t)$. In the Eulerian coordinates, the corresponding deformation tensor $\mathcal{F}(\mathbf{x},t)$ is defined as $\mathcal{F}(\mathbf{x}(X,t),t) = \tilde{\mathcal{F}}(X,t)$. Using the chain rule, one obtains (3), which stands for $\partial_t(\mathcal{F}_{ij}) + u_k\partial_{x_k}\mathcal{F}_{ij} = \partial_{x_k}u_i\mathcal{F}_{kj}$, $1\le i,j\le n$. Denote $(\nabla\cdot\mathcal{F})_j = \partial_{x_i}\mathcal{F}_{ij}$. Taking divergence of both sides of (3) and using $\nabla\cdot\mathbf{u}=0$, one gets the transport equation for $\nabla\cdot\mathcal{F}$ as

$$(\nabla\cdot\mathcal{F})_t + \mathbf{u}\cdot\nabla(\nabla\cdot\mathcal{F}) = 0. \tag{4}$$

We assume naturally the initial datum $\mathcal{F}_0 = I$, where I is the identity matrix. Thus $\nabla\cdot\mathcal{F}_0 = 0$ and $\det\mathcal{F}_0 = 1$. Then (4) yields

$$\nabla\cdot\mathcal{F} = 0, \quad \forall\, t\ge 0, \tag{5}$$

which implies that in the 2-D case ($n=2$), there exists a vector function $\boldsymbol{\phi} = (\phi_1,\phi_2)$ such that

$$\begin{aligned}\mathcal{F} &= \nabla^{\perp}\boldsymbol{\phi} = \begin{pmatrix} -\partial_{x_2}\boldsymbol{\phi} \\ \partial_{x_1}\boldsymbol{\phi}\end{pmatrix} \\ &= \begin{pmatrix} -\partial_{x_2}\phi_1 & -\partial_{x_2}\phi_2 \\ \partial_{x_1}\phi_1 & \partial_{x_1}\phi_2\end{pmatrix}.\end{aligned} \tag{6}$$

Therefore, (1)-(3) in $\mathbb{R}^2$ can be transformed into an equivalent form as follows (see [1]):

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} - \lambda\nabla\cdot(\nabla\boldsymbol{\phi}\otimes\nabla\boldsymbol{\phi}), \tag{7}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{8}$$

$$\boldsymbol{\phi}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\phi} = 0, \tag{9}$$

where $P = p - |\partial_{x_1}\boldsymbol{\phi}|^2 - |\partial_{x_2}\boldsymbol{\phi}|^2$ and $\nabla\boldsymbol{\phi}\otimes\nabla\boldsymbol{\phi}$ is a 2×2 matrix whose (i,j)-th entry is $\partial_{x_i}\boldsymbol{\phi}\cdot\partial_{x_j}\boldsymbol{\phi}$ for $1\le i,j\le 2$.

Due to $\det\mathcal{F}_0 = 1$ and (6), we assume the initial datum $\boldsymbol{\phi}_0$ of system (7)-(9) satisfies $\det(\nabla^{\perp}\boldsymbol{\phi}_0) = 1$. It was proved in [1] that the system (7)-(9) has global classical solutions on the whole plane $\mathbb{R}^2$, provided that the initial data $(\mathbf{u}_0,\boldsymbol{\phi}_0)$ satisfies the following assumptions:

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(A1) $\mathbf{u}_0$ is near zero;
(A2) $\det(\nabla^{\perp}\boldsymbol{\phi}_0) = 1$, $\boldsymbol{\phi}_0$ is close to the vector $\mathbf{a} = (-x_2, x_1)$;
(A3) $\mathbf{u}_0 \in H^k(\mathbb{R}^2)$ and $\nabla^{\perp}(\boldsymbol{\phi}_0 - \mathbf{a}) \in H^k(\mathbb{R}^2)$, $k \in \mathbb{N}$, $k \geq 2$.
Furthermore, the authors in [2] proved that the system (1)-(3) has global smooth solutions in the whole space $\mathbb{R}^n$ ($n = 2, 3$) if $(\mathbf{u}_0, \mathcal{F}_0)$ satisfies similar conditions to (A1)-(A3). Specifically, the three assumptions are:
$(A1)'$ $\mathbf{u}_0$ is near zero;
$(A2)'$ $\det \mathcal{F}_0 = 1$, $\nabla \cdot \mathcal{F}_0 = 0$, $\mathcal{F}_0$ is near the identity matrix;
$(A3)'$ $\mathbf{u}_0, \mathcal{F}_0 - I \in H^k(\mathbb{R}^2)$, $k \in \mathbb{N}$, $k \geq 2$.
There are no results for global existence if the magnitude of $\mathbf{u}_0$ is large.

Due to the above result in [1], initial data of a blow up solution to (7)-(9) could not satisfy every condition of (A1)-(A3). In the first part of this letter, we construct such a blow up solution of (7)-(9) that its initial data satisfies (A2), but violates (A1) and (A3). We use the method of separation of variables. Assume

$$
\begin{aligned}
&u_1(\mathbf{x}, t) = x_1 f(t),\ u_2(\mathbf{x}, t) = -x_2 f(t), && (10)\\
&P(\mathbf{x}, t) = P_1(\mathbf{x}) g(t), && (11)\\
&\phi_1(\mathbf{x}, t) = -x_2 h_1(t) + x_1 g_1(t), && (12)\\
&\phi_2(\mathbf{x}, t) = x_1 h_2(t) - x_2 g_2(t). && (13)
\end{aligned}
$$

Then the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ holds. Since $\mathcal{F}_0 = \nabla^{\perp}\boldsymbol{\phi}_0$ and $\mathcal{F}_0$ is the identity matrix, we have

$$
\begin{aligned}
\nabla^{\perp}\boldsymbol{\phi}_0 &= \left.\begin{pmatrix} -\partial_{x_2}\phi_1 & -\partial_{x_2}\phi_2 \\ \partial_{x_1}\phi_1 & \partial_{x_1}\phi_2 \end{pmatrix}\right|_{t=0} \\
&= \left.\begin{pmatrix} h_1(t) & g_2(t) \\ g_1(t) & h_2(t) \end{pmatrix}\right|_{t=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (14)
\end{aligned}
$$

In addition, we have $\Delta \mathbf{u} = 0$ and $\nabla \cdot (\nabla\boldsymbol{\phi} \otimes \nabla\boldsymbol{\phi}) = 0$.

Substituting (10)-(13) into (7), we get

$$
\begin{aligned}
x_1 f'(t) + x_1 f^2(t) + \frac{\partial P_1(\mathbf{x})}{\partial x_1} g(t) &= 0, \\
-x_2 f'(t) + x_2 f^2(t) + \frac{\partial P_1(\mathbf{x})}{\partial x_2} g(t) &= 0.
\end{aligned}
$$

By separation of variables, we obtain

$$
\begin{aligned}
\frac{f'(t) + f^2(t)}{g(t)} &= \frac{-1}{x_1}\frac{\partial P_1(\mathbf{x})}{\partial x_1} = -\alpha, && (15)\\
\frac{f'(t) - f^2(t)}{g(t)} &= \frac{1}{x_2}\frac{\partial P_1(\mathbf{x})}{\partial x_2} = -\beta, && (16)
\end{aligned}
$$

where α, β are constants. Obviously, $P_1(\mathbf{x})$ has the form of $P_1(\mathbf{x}) = \frac{1}{2}(\alpha x_1^2 - \beta x_2^2)$ up to a constant. Suppose $f(0) = f_0$. Solving Eqs. (15) and (16), we obtain

$$
\begin{aligned}
f(t) &= \frac{f_0}{1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t}, && (17)\\
g(t) &= \frac{2}{\beta - \alpha} f^2(t) \\
&= \frac{2 f_0^2}{(\beta - \alpha)\left(1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right)^2}. && (18)
\end{aligned}
$$

Substituting (10)-(13) into (9), we obtain

$$
\begin{aligned}
x_1(g_1'(t) + f(t) g_1(t)) - x_2(h_1'(t) - f(t) h_1(t)) = 0, \\
x_1(h_2'(t) + f(t) h_2(t)) - x_2(g_2'(t) - f(t) g_2(t)) = 0,
\end{aligned}
$$

for arbitrary x_1 and x_2. Therefore,

$$g_1'(t) + f(t)g_1(t) = 0, \quad h_1'(t) - f(t)h_1(t) = 0,$$
$$h_2'(t) + f(t)h_2(t) = 0, \quad g_2'(t) - f(t)g_2(t) = 0,$$

which are subject to the initial condition (14). Hence, we get

$$\begin{aligned}
g_1(t) &= g_1(0)e^{-\int_0^t f(s)ds} \equiv 0, \\
g_2(t) &= g_2(0)e^{\int_0^t f(s)ds} \equiv 0, \\
h_1(t) &= h_1(0)e^{\int_0^t f(s)ds} = e^{\int_0^t f(s)ds} \\
&= \left|1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right|^{\frac{\beta-\alpha}{\alpha+\beta}}, \qquad (19) \\
h_2(t) &= h_2(0)e^{-\int_0^t f(s)ds} = e^{-\int_0^t f(s)ds}, \\
&= \left|1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right|^{\frac{\alpha-\beta}{\alpha+\beta}}. \qquad (20)
\end{aligned}$$

Finally, substituting (17), (18), (19) and (20) into (10)-(13), we find the following explicit solutions of (7)-(9) with initial datum $\boldsymbol{\phi}_0$ satisfying (A2):

$$\begin{cases}
\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} \frac{x_1 f_0}{1-\frac{\alpha+\beta}{\alpha-\beta}f_0 t} \\ \frac{-x_2 f_0}{1-\frac{\alpha+\beta}{\alpha-\beta}f_0 t} \end{pmatrix}, \\
P(\mathbf{x}, t) = \frac{\left(\alpha x_1^2 - \beta x_2^2\right) f_0^2}{(\beta-\alpha)\left(1-\frac{\alpha+\beta}{\alpha-\beta}f_0 t\right)^2}, \\
\boldsymbol{\phi}(\mathbf{x}, t) = \begin{pmatrix} -x_2 \left|1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right|^{\frac{\beta-\alpha}{\alpha+\beta}} \\ x_1 \left|1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right|^{\frac{\alpha-\beta}{\alpha+\beta}} \end{pmatrix},
\end{cases} \qquad (21)$$

where f_0, α and β are constants.

With the use of (6) and (21), the corresponding deformation tensor $\mathcal{F}$ becomes

$$\begin{aligned}
\mathcal{F} &= \mathcal{F}(t) \\
&= \begin{pmatrix} \left|1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right|^{\frac{\beta-\alpha}{\alpha+\beta}} & 0 \\ 0 & \left|1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right|^{\frac{\alpha-\beta}{\alpha+\beta}} \end{pmatrix}.
\end{aligned}$$

If $\frac{\alpha+\beta}{\alpha-\beta} f_0 > 0$, $\alpha + \beta \neq 0$ and $\alpha - \beta \neq 0$, (21) will blow up at time $t^* = \frac{\alpha-\beta}{(\alpha+\beta)f_0}$. Recall the conditions (A1)-(A3) which guarantee the global existence of smooth solutions, here (A1) and (A3) are violated in (21). As $t \to t^*$, $1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t \to 0$, one diagonal element of $\mathcal{F}$ tends to zero, while the other tends to infinity. This means the viscoelastic fluid is squeezed in one spatial direction, but stretched in the other direction.

Below we will construct some blow up solutions to the following coupled Navier-Stokes/Allen-Cahn system :

$$\begin{aligned}
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P &= \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \phi \otimes \nabla \phi), \qquad (22) \\
\nabla \cdot \mathbf{u} &= 0, \qquad (23) \\
\phi_t + (\mathbf{u} \cdot \nabla)\phi &= \gamma(\Delta \phi - f(\phi)). \qquad (24)
\end{aligned}$$

(24) is the advective Allen-Cahn equation. If it is replaced with the advective Cahn-Hilliard equation:

$$\phi_t + (\mathbf{u} \cdot \nabla)\phi = -\gamma \Delta(\Delta \phi - f(\phi)), \qquad (25)$$

then the system (22), (23), (25) is a coupled Navier-Stokes/Cahn-Hilliard system. If $\gamma = 0$ and the scalar function ϕ in (24)/(25) is taken as a vector function $\boldsymbol{\phi} = (\phi_1, \phi_2)$, then we formally arrive at (7)-(9).

The system (22)-(24)/(25) are two types of Navier-Stokes/phase-field model, describing the motion of incompressible viscous two-phase fluids (see, for example, [3–5]). The two fluids are separated by a thin interface of width $\varepsilon > 0$, which is a small constant. $\mathbf{u}$ represents the velocity field of the mixture, P is the pressure, and ϕ is the phase function, taking the value 1 in one bulk phase and -1 in the other. In the interfacial region, ϕ varies rapidly and smoothly. $\nabla \phi \otimes \nabla \phi$ denotes the induced elastic stress, which is a 2×2 matrix whose (i, j)-th entry is $\frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j}$ for $1 \leq i, j \leq 2$.

$f(\phi) = \frac{1}{\varepsilon^2}(\phi^3 - \phi)$. ν, λ and γ are positive constants, which denote the kinetic viscosity constant, the mixing energy density and the mobility, respectively.

We want to find blow up solutions of (22)-(24)/(25) in $\mathbb{R}^2$, which are based on solutions of incompressible Navier-Stokes equations:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\Delta\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{26}$$

An explicit blow up solution of (26) in $\mathbb{R}^2$ is constructed as follows (see [6]):

$$\begin{cases} u_1(\mathbf{x},t) = & \frac{1}{\sqrt{T-t}}\left(-1 + c_1\exp\left(\frac{s^2}{8\nu(T-t)}\right.\right. \\ & \left.\left. -\frac{s}{\nu\sqrt{T-t}} + c_3\right)\right), \\ u_2(\mathbf{x},t) = & \frac{1}{\sqrt{T-t}}\left(-1 - c_1\exp\left(\frac{s^2}{8\nu(T-t)}\right.\right. \\ & \left.\left. -\frac{s}{\nu\sqrt{T-t}} + c_3\right)\right), \\ p(\mathbf{x},t) = & \frac{s}{2(T-t)^{3/2}} + \frac{c_2}{T-t}, \end{cases} \tag{27}$$

where $s = x_1 + x_2$, T is a positive constant and c_i $(i = 1, 2, 3)$ are constants.

We suppose

$$\phi(\mathbf{x},t) = \Phi(s + f(t)), \tag{28}$$

where $\Phi(\cdot)$ and $f(\cdot)$ are non-constant C^1 functions to be determined. Then $\partial_{x_1}\phi = \partial_{x_2}\phi = \Phi'$, and $\nabla\cdot(\nabla\phi\times\nabla\phi)$ can be rewritten as $2\nabla(\Phi')^2$. Hence, comparing (26) with (22), we observe that u_1, u_2 in (27) and $P = p - 2\lambda(\Phi')^2$ satisfy (22).

Next we consider (24) in a simple case $\gamma = 0$. Substituting (28) into

$$\phi_t + (\mathbf{u}\cdot\nabla)\phi = 0, \tag{29}$$

and using $\Phi' \neq 0$, we get

$$f'(t) - \frac{2}{\sqrt{T-t}} = 0. \tag{30}$$

Solving (30), we have $f(t) = -4\sqrt{T-t} + C$, where C is a constant. Therefore, $\phi(\mathbf{x},t) = \Phi(s - 4\sqrt{T-t} + C)$ is a solution to (29).

When $\gamma > 0$, note that $\tanh(\frac{x_1+x_2}{2\varepsilon})$ is a solution to $\Delta\phi - f(\phi) = 0$. Hence, taking $\Phi(\cdot) = \tanh(\frac{\cdot}{2\varepsilon})$, namely, $\phi(\mathbf{x},t) = \tanh(\frac{x_1+x_2-4\sqrt{T-t}+C}{2\varepsilon})$, we get a solution to (24)/(25).

Finally, we obtain a blow up solution of (22)-(24)/(25) in $\mathbb{R}^2$:

$$\begin{cases} u_1(\mathbf{x},t) = & \frac{1}{\sqrt{T-t}}\left(-1 + c_1\exp\left(\frac{s^2}{8\nu(T-t)}\right.\right. \\ & \left.\left. -\frac{s}{\nu\sqrt{T-t}} + c_3\right)\right), \\ u_2(\mathbf{x},t) = & \frac{1}{\sqrt{T-t}}\left(-1 - c_1\exp\left(\frac{s^2}{8\nu(T-t)}\right.\right. \\ & \left.\left. -\frac{s}{\nu\sqrt{T-t}} + c_3\right)\right), \\ \phi(\mathbf{x},t) = & \tanh(\frac{s-4\sqrt{T-t}+c_4}{2\varepsilon}), \\ P(\mathbf{x},t) = & \frac{s}{2(T-t)^{3/2}} - \frac{\lambda}{2\varepsilon^2}\cosh^{-4}(\frac{s-4\sqrt{T-t}}{2\varepsilon}) \\ & +\frac{c_2}{T-t}. \end{cases}$$

The above solution can be easily generalized to the 3-D case:

$$\begin{cases} u_1(\mathbf{x},t) = & \frac{1}{\sqrt{T-t}}\left(-1+c_1\exp\left(\frac{s^2}{12\nu(T-t)}\right.\right. \\ & \left.\left.-\frac{s}{\nu\sqrt{T-t}}+c_4\right)\right), \\ u_2(\mathbf{x},t) = & \frac{1}{\sqrt{T-t}}\left(-1+c_2\exp\left(\frac{s^2}{12\nu(T-t)}\right.\right. \\ & \left.\left.-\frac{s}{\nu\sqrt{T-t}}+c_4\right)\right), \\ u_3(\mathbf{x},t) = & \frac{1}{\sqrt{T-t}}\left(-1-(c_1+c_2)\exp\left(\frac{s^2}{12\nu(T-t)}\right.\right. \\ & \left.\left.-\frac{s}{\nu\sqrt{T-t}}+c_4\right)\right), \\ \phi(\mathbf{x},t) = & \tanh(\frac{s-6\sqrt{T-t}+c_5}{\sqrt{6}\varepsilon}), \\ P(\mathbf{x},t) = & \frac{s}{2(T-t)^{3/2}}-\frac{\lambda}{2\varepsilon^2}\cosh^{-4}(\frac{s-6\sqrt{T-t}}{\sqrt{6}\varepsilon}) \\ & +\frac{c_3}{T-t}. \end{cases}$$

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